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Solution of the Viscous Transonic Equation for Flow Past a Wavy Wall

Interim Technical Report

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May 1970

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SOLUTION OF THE VISCOUS TRANSONIC EQUATION
FOR FLOW PAST A WAVY WALL

by Martin Sichel

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May 1970

SUMMARY

The viscous transonic small disturbance equation has been recast into a form which displays the role of the usual inviscid transonic similarity parameter and of a newly defined viscous transonic similarity parameter in the two dimensional flow past slender bodies. Using the approximate method of Hosokawa the solution for viscous transonic flow past a wavy wall has been obtained in analytical form, and displays the role of the above similarity parameters. The influence of the Reynolds number and free stream Mach number on the supersonic pockets which arise at the wall surface has been investigated.

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I. INTRODUCTION

Regions of supersonic flow arise near the surfaces of airfoils and nozzles as the free stream Mach number approaches unity. Such supersonic pockets are usually, though not always (Nieuwland and Spee, 1968) terminated by a weak shock wave. Analysis of the flow within and near such pockets, particularly in the neighborhood of the terminating shock is a crucial problem of transonic aerodynamics. A particular difficulty is that the pressure rise across the terminating shock wave is not the same as that obtained from the Rankine-Hugoniot equations due to the interaction between the shock and the boundary layer at the surface and also due to the fact that the weak terminating shock may be sufficiently thick that the shock structure is no longer one dimensional (Sichel, 1968).

To gain some insight into this problem Hosokawa (1960a,b), using an approximate method, obtained solutions of the inviscid transonic equation for flow past a wavy wall. With increasing free stream Mach number supersonic pockets do appear at the surface of the wavy wall, and Hosokawa's analysis always requires that such supersonic regions be terminated by a shock like discontinuity. Hosokawa's method was applied by Sichel and Yin (1969) to obtain solutions of the viscous transonic (or V-T) equation for wavy wall flow. The V-T equation, which is discussed in a survey by Sichel (1968), corrects the inviscid transonic equation by an additional term representing the influence

of compressive or longitudinal viscosity, but shear viscosity is still neglected. Thus the V-T equation makes it possible to analyze weak shock waves with a two dimensional internal structure. Sichel and Yin's results could only be obtained by numerical integration and hence were limited to a single Reynolds number and only several values of free stream Mach number.

In the present paper the V-T equation is first recast into a form which displays the role of the usual inviscid transonic similarity parameter, and a newly defined viscous transonic similarity parameter in transonic flows about bodies in general. Then, using the method of Hosokawa, the solution for V-T flow past a wavy wall is obtained in analytical form so that the influence of Reynolds number and free stream Mach number on the supersonic pockets which arise can be explored in detail.

II. THE VISCOUS TRANSONIC EQUATION AND SIMILARITY

The V-T small disturbance equation has been derived (Sichel, 1968) from the Navier-Stokes equations by perturbing with respect to a uniform sonic flow. But, a more general equation is obtained if perturbations are with respect to some uniform free stream velocity $\overline{U}$ near the sonic value but not necessarily equal to it. Consider two dimensional flow past a body, whose surface is given by

$$\overline{y} - \overline{y}_w(\overline{x}) = \overline{S} f(\overline{x}/\overline{L}) \tag{1}$$

as shown in Fig. 1. Barred quantities are dimensional. $\overline{S}$ is the maximum thickness, and the body shape function $f(\overline{x}/\overline{L}) \sim O(1)$. Then using $\overline{U}$, $\overline{\rho}_\infty$, $\overline{\rho}_\infty \overline{U}^2$ as reference velocity, density and pressure; and the body length, $\overline{L}$, as reference length, the expansions

$$u = \frac{\overline{u}}{\overline{U}} = 1 + \epsilon u^{(1)} + \dots \qquad\qquad p = \frac{1}{\gamma M_\infty^2} + \epsilon p^{(1)} + \dots$$

$$\tag{2}$$

$$\rho = \frac{\rho}{\overline{\rho}_\infty} = 1 + \epsilon \rho^{(1)} + \dots \qquad\qquad u = \frac{\overline{v}}{\overline{U}} = \lambda \epsilon v^{(1)}$$

and the coordinate stretching

$$x = \frac{\overline{x}}{\overline{L}} \qquad ; \qquad y = \lambda \frac{\overline{y}}{\overline{L}} \tag{3}$$

when introduced in the Navier Stokes equations, yield the following equations for $u^{(1)}$ and $v^{(1)}$:

$$\frac{\bar{\nu}''}{\overline{U}\,\overline{L}\,\epsilon}\left(1 + \frac{\gamma - 1}{Pr''}\right)u^{(1)}_{xx} + \frac{(1 - M_\infty^2)}{\epsilon}u^{(1)}_{x}$$

$$- (\gamma + 1)\,M_\infty^2\,u^{(1)}\,u^{(1)}_{x} + \frac{\lambda^2}{\epsilon}v^{(1)}_{y} = 0 \tag{4}$$

$$u^{(1)}_{y} = v^{(1)}_{x} \tag{5}$$

The stretch factor λ takes into account the difference in the characteristic x and y dimensions which arises in transonic flow while $u^{(1)}$, $\rho^{(1)}$, $p^{(1)}$, and $v^{(1)}$ are presumed to be O(1), and the expansion parameter $\epsilon \ll 1$. $\bar{\nu}''$ is the compressive or longitudinal viscosity and Pr'' is the longitudinal Prandtl number. Since the flow is irrotational to the present order of approximation introduction of a potential Φ reduces Eqs. (4) and (5) to

$$\frac{\bar{\nu}''}{\overline{U}\,\overline{L}\,\epsilon}\left(1 + \frac{\gamma - 1}{Pr''}\right)\Phi_{xxx} + \frac{(1 - M_\infty^2)}{\epsilon}\Phi_{xx} - (\gamma + 1)\,M_\infty^2\,\Phi_x\,\Phi_{xx} + \frac{\lambda^2}{\epsilon}\Phi_{yy} = 0 \tag{6}$$

To the present order of approximation the boundary conditions will be that

$$\lambda\,\epsilon\,v^{(1)}(x, 0) = \lambda\,\epsilon\,\Phi_y(x, 0) = \frac{d\bar{y}_w}{d\bar{x}} = S\,\frac{df}{dx} \tag{7}$$

where $S = \overline{S}/\overline{L}$, and that $u^{(1)}$ and $v^{(1)}$ must remain finite as $y \to \infty$. At the same time the pressure coefficient C_p is given by

$$C_p = \frac{\bar{p} - \bar{p}_\infty}{\frac{1}{2} \bar{\rho}_\infty \bar{U}^2} = - 2\epsilon \, \Phi_x$$

Following Ferrari and Tricomi's (1968) discussion of inviscid transonic similarity let

$$\lambda\epsilon = S \tag{8}$$

since both $v^{(1)}(x, 0)$ and $f'(x)$ should be O(1).

Using Eq. (8) the V-T equations can then be written in the form

$$\frac{\bar{\nu}'' \epsilon^2}{\bar{U} \, L \, S^2} \left(1 + \frac{\gamma - 1}{Pr''}\right) \Phi_{xxx} + \frac{(1 - M_\infty^2) \, \epsilon^2}{S^2} \Phi_{xxx}$$

$$- \frac{(\gamma + 1) \, M_\infty^2 \, \epsilon^3}{S^2} \Phi_x \, \Phi_{xx} + \Phi_{yy} = 0 \tag{9}$$

Now, in V-T flow the four terms of Eq. (9) should be of the same order, and if the expansion and stretching of Eqs. (2) and (3) are appropriate, the Φ_{xxx}, Φ_{xx}, Φ_x, $\Phi_{yy} \sim$ O(1), and it becomes convenient to choose ϵ so that

$$\frac{(\gamma + 1) \, M_\infty^2 \, \epsilon^3}{S^2} = 1 \quad \text{or} \quad \epsilon = \frac{S^{2/3}}{(\gamma + 1)^{1/3} \, M_\infty^{2/3}} \tag{10}$$

The coefficient of Φ_{xx} then becomes

$$\frac{(1 - M_\infty^2) \epsilon^2}{S^2} = \frac{(1 - M_\infty^2)}{S^{2/3} \, M_\infty^{4/3} \, (\gamma + 1)^{2/3}} = X_\infty \tag{11}$$

where χ_∞ is now the usual inviscid transonic similarity parameter (Ferrari and Tricomi, 1968). The coefficient of the viscous term of Eq. (9) becomes

$$\frac{\overline{\nu}'' \epsilon^2}{\overline{U}\,\overline{L}\,S^2}\left(1 + \frac{\gamma - 1}{Pr''}\right) = \frac{\overline{\nu}''}{\overline{U}\,\overline{L}\,S^{2/3}\,M_\infty^{4/3}\,(\gamma + 1)^{2/3}}\left(1 + \frac{\gamma - 1}{Pr''}\right) = \chi_v \qquad (12)$$

where χ_v is now a viscous-transonic similarity parameter. The V-T equation can thus be written in the form

$$\chi_v\,\Phi_{xxx} + \chi_\infty\,\Phi_{xx} - \Phi_x\,\Phi_{xx} + \Phi_{yy} = 0 \qquad (13)$$

with the boundary conditions

$$\Phi_y(x, 0) = f'(x) \quad ; \quad \Phi_y, \Phi_x \quad \text{bounded as} \quad |y| \to \infty, \ |x| \to \infty \qquad (14)$$

The pressure coefficient will be

$$C_p = -\frac{2S^{2/3}}{(\gamma + 1)^{1/3}\,M_\infty^{2/3}}\,\Phi_x(x, y; \chi_\infty, \chi_v) \qquad (15)$$

Equations (13), (14), and (15), provide the basis for viscous-transonic similarity rules.

Since $\overline{U}\,\overline{L}/\overline{\nu}''$ is a Reynolds number, Re, based on compressive viscosity, the newly defined V-T similarity parameter, χ_v can also be expressed in the form

$$\chi_v = \frac{1}{Re\,S^{2/3}\,M_\infty^{4/3}\,(\gamma + 1)^{2/3}}\left(1 + \frac{\gamma - 1}{Pr''}\right) \qquad (16)$$

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Thus χ_v varies inversely with the Reynolds number, and the coefficient of the viscous term in Eq. (13) will become very small for large Re. Letting

$$\bar{\eta} = \frac{\bar{\nu}''}{\bar{U}\, S^{2/3}\, M_\infty^{4/3}\, (\gamma + 1)^{2/3}} \left(1 + \frac{\gamma - 1}{Pr''}\right) \tag{17}$$

the V-T similarity parameter χ_v becomes

$$\chi_v = \bar{\eta}/\bar{L} \tag{18}$$

and $\bar{\eta}$ is a length of the same order as the thickness of a weak shock wave. Finally, in terms of a viscous length $\bar{L}_v = \bar{\nu}''/\bar{a}_\infty$,

$$\chi_v = \frac{(\bar{L}_v/\bar{L})\left(1 + \frac{\gamma - 1}{Pr''}\right)}{S^{2/3}\, M_\infty^{7/3}\, (\gamma + 1)^{2/3}} \tag{19}$$

Since $\bar{L}_v$ is of the order of the mean free path, the ratio $\bar{L}_v/\bar{L}$ is clearly a Knudsen number.

It is readily shown that at the sonic point

$$u^{(1)} = u^{(1)}_c = \frac{(1 - M_\infty^2)}{S^{2/3}\, M_\infty^{4/3}\, (\gamma + 1)^{2/3}} = \chi_\infty \tag{20}$$

that is the critical value of the expansion coefficient $u^{(1)}$ equals the inviscid transonic similarity parameter.

The variation of χ_∞ with S and M_∞ is shown in Fig. 2(a), while Fig. 2(b) shows the variation of χ_v with the Reynolds number $\overline{U}\,\overline{L}/\nu''$ for different values of thickness ratio S, and also shows the variation of $\overline{L}_v/\overline{L}$ with Re. Clearly, the use of the continuum theory is questionable when $\overline{L}_v/\overline{L} > 0.1$ or when Re < 10 in the present case with $M_\infty \cong 1.0$. With Re > 100, $\chi_v \ll 1$, except for extremely small values of the thickness ratio S, and then solution of the V-T equation (13) becomes a singular perturbation problem.

Equation (15) implies the similarity rule that the pressure coefficient C_p, varies as $S^{2/3}$ for fixed χ_∞, χ_v, at least to first order in $(1 - M_\infty^2)$ or ϵ. For geometrically similar bodies it is necessary to know the influence of the physical variables $\overline{L}$, M_∞, $\overline{S}$, and $\overline{L}_v$ upon the similarity parameters χ_∞ and χ_v to assess the significance of the V-T similarity rule implied by Eqs. (13), (14), and (15); hence the effects of these variations have been summarized in Table I below. Case 1 in Table I represents flow with constant free stream density, temperature, and velocity, but with variable characteristic length $\overline{L}$, which might, for instance, be the airfoil chord. Then even though the inviscid transonic similarity parameter remains constant, the viscous parameter χ_v varies inversely with $\overline{L}$. Flow past a fixed object with constant $\overline{\rho}_\infty$, $\overline{T}_\infty$ but variable M_∞ is represented by Case 2, and then χ_v is fixed but the inviscid parameter χ_∞ is variable. Cases 3 and 4 show the effects of varying body thickness $\overline{S}$, and of varying $\overline{\rho}_\infty$ and $\overline{T}_\infty$ and hence the viscous length $\overline{L}_v$ upon the similarity parameters. Case 5 shows the variation of C_p when parameters are adjusted to keep χ_∞ and χ_v fixed.

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Table I. Relation Between Physical Variables and Similarity Parameters in V-T Flow.

Case No.	Physical Variables		Similarity Parameters	
	Fixed	Variable	Fixed	Variable
1.	$M_\infty, S, \overline{L}_v$	$\overline{L}$	χ_∞	$\chi_v \propto \overline{L}^{-1}$
2.	$\overline{S}, \overline{L}, \overline{L}_v$	M_∞	χ_v	$\chi_\infty \propto (1 - M_\infty^2)$
3.	$\overline{L}, \overline{L}_v, M_\infty$	$\overline{S}$	χ_∞ / χ_v	$\chi_\infty \propto \overline{S}^{-2/3}$
				$\chi_v \propto \overline{S}^{-2/3}$
4.	$\overline{S}, \overline{L}, M_\infty$	$\overline{L}_v$	χ_∞	$\chi_v \propto \overline{L}_v$
5.	$\overline{L}_v$	$\overline{S}, \overline{L}, M_\infty$	χ_∞, χ_v	$C_p \propto S^{2/3}$
		$1 - M_\infty^2 \propto S^{2/3}$		
		$\overline{L} \propto \overline{S}^{-2}$		

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III. WAVY WALL PROBLEM — FORMULATION

The problem of flow past a wavy wall provides a means of examining the effects of the parameters χ_∞ and χ_v on V-T flow since, using the method of Hosokawa (1960), it is possible to obtain approximate solutions of the V-T equation in analytical form.

With the ordinate of the wall given by

$$\overline{y}_w = \overline{S} \sin 2\pi \frac{\overline{x}}{\overline{\ell}} \tag{21}$$

where $\overline{\ell}$ is the wavelength, the boundary condition (Eq. (14)) becomes

$$v^{(1)} = \Phi_y(x, 0) = \omega \cos \omega x \tag{22}$$

where $\omega = 2\pi \overline{L}/\overline{\ell}$. The factor ω determines the choice of the characteristic length $\overline{L}$. Thus letting $\omega = 1$ implies that $\overline{L} = \overline{\ell}/2\pi$, and, as will become evident later, is the most convenient choice. In the present formulation the influence of $\overline{S}$ and $\overline{\ell}$ enters only through the similarity parameters χ_∞ and χ_v in Eq. (13), with the choice of ω a matter of computational convenience, in contrast to the previous formulation (Sichel and Yin, 1969) where the parameters of the problem occur in both the equation and the boundary conditions.

Following Hosokawa (1960b) the potential Φ is split into two parts

$$\Phi = \phi + g \tag{23}$$

such that ϕ satisfies a linearized V-T equation with the acceleration ϕ_{xx} in the

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nonlinear term replaced by a constant K so that

$$\chi_v \phi_{xxx} + \chi_\infty \phi_{xx} - K \phi_x + \phi_{yy} = 0 \qquad (24)$$

ϕ is made to satisfy the wavy wall boundary condition (Eq. (22)) so that

$$\phi_y(x, 0) = \omega \cos \omega x = \text{Re} \, (\omega \, e^{i\omega x}) \qquad (25)$$

The function g is a nonlinear correction which in view of Eqs. (13), (24), and (25) must satisfy

$$\chi_v g_{xxx} + \chi_\infty g_{xx} + (K - \phi_{xx}) \phi_x - (g_x \phi_x)_x - g_x g_{xx} + g_{yy} = 0 \qquad (26)$$

$$g_y(x, 0) = 0$$

A key assumption in the approximate analysis, which is discussed in detail by Hosokawa (1960b) and by Sichel and Yin (1969) is that

$$g_{yy}(x, 0) = 0 \qquad (27)$$

so that Eq. (26) can be treated as an ordinary differential equation for g at the wall, y = 0. Following Hosokawa (1960b) the constant K is chosen as $\phi_{xx}(x, 0)$ at the accelerating sonic point, $\phi_x(x, 0) = \chi_\infty$, in the linear solution where K > 0. This procedure yields the equations

$$\phi_x \, (x^*, 0; \, \chi_\infty, \chi_v) = \chi_\infty$$

$$\phi_{xx} \, (x^*, 0; \, \chi_\infty, \chi_v) = K \qquad (28)$$

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for K and the sonic point $x = x^*$. The justification for the choice of K is discussed by both Hosokawa (1960b) and Sichel and Yin (1969). Equations (23)-(28) present the basic elements of Hosokawa's approximate method.

Assuming that ϕ is of the form

$$\phi = \text{Re} \left[F(y) \, e^{i\omega x} \right]$$

the solution of Eq. (24) with boundary condition (Eq. (25)) is readily shown to be

$$\phi = \text{Re} \left[- (\omega/m^{1/2}) \exp\left(- m^{1/2} y \cos \frac{\beta}{2}\right) \exp i \left(\omega x - m^{1/2} y \sin \frac{\beta}{2} - \frac{\beta}{2}\right) \right] \quad (29)$$

where

$$m = \left[\chi_\infty^2 \omega^4 + \left(\omega^3 \chi_v + \omega K\right)^2 \right]^{1/2} \quad (30)$$

$$\beta = \arctan \frac{\omega^3 \chi_v + \omega K}{\chi_\infty \omega^2} \quad ; \quad 0 \leq \beta \leq \pi$$

Letting $K = \chi_v = 0$ reduces the linearized equation (24) to that for inviscid linearized subsonic or supersonic flow, and the solution (29) reduces to the well known linearized subsonic or supersonic wavy wall solution.

Equation (26) for $g(x, 0)$ can be integrated once to yield

$$\chi_v g_{xx} + \chi_\infty g_x - \frac{1}{2} (\phi_x + g_x)^2 + K\phi = A_1 \quad (31)$$

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where A_1 is a constant of integration. Introduction of the variable

$$\zeta(x) = g_x(x, 0) + \phi_x(x, 0) - \chi_\infty = u^{(1)}(x, 0) - u^{(1)}_c \tag{32}$$

for the deviation of $u^{(1)}(x, 0)$, the velocity at the wall, from the critical value and use of the linearized solution for ϕ then yields the following ordinary differential equation for ζ:

$$\zeta' - \frac{1}{2\chi_v} \zeta^2 = -\frac{\chi_\infty}{2\chi_v} + \frac{A_1}{\chi_v} + \frac{m^{1/2}}{\chi_v} \cos\left(\omega x + \frac{\beta}{2} - \frac{\pi}{2}\right) \tag{33}$$

Equation (33) is a Riccati equation, which with the transformation

$$\xi = \frac{\omega}{2} x + \frac{\beta}{4} + \frac{\pi}{4} \quad ;$$

$$\tag{34}$$

$$\zeta = -\omega\chi_v \cdot \frac{1}{T} \frac{dT}{d\xi}$$

changes to the following second order linear equation for T:

$$T'' + (\hat{a} - 2q \cos 2\xi)\, T = 0 \tag{35}$$

with

$$\hat{a} = \frac{2A_1 - \chi_\infty^2}{\omega^2 \chi_v^2} \quad ; \quad q = \frac{m^{1/2}}{\omega^2 \chi_v^2}$$

Equation (35) is the Mathieu equation whose properties are well known (McLachlan, 1947). The V-T wavy wall flow has thus been reduced to the problem of solving the Mathieu equation and is considered below.

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IV.   RELATION BETWEEN χ_v, χ_∞, AND THE PARAMETERS OF THE MATHIEU EQUATION

The wavy wall solution depends upon the similarity parameters χ_v and χ_∞ through their influence on $\hat{a}$ and q in the Mathieu equation (35).   Therefore it is necessary to solve Eq. (28), which relates m and the angle β to χ_v and χ_∞.

Introducing the solution for ϕ, Eq. (28) becomes

$$\frac{\omega^2}{m^{1/2}} \sin\left(\omega x^* - \frac{\beta}{2}\right) = \chi_\infty$$

$$\frac{\omega^3}{m^{1/2}} \cos\left(\omega x^* - \frac{\beta}{2}\right) = K \qquad (36)$$

In solving Eq. (36) it is more convenient to deal with the variable

$$\eta = \tan\beta = \frac{\omega\chi_v + (K/\omega)}{\chi_\infty} \qquad (37)$$

so that

$$m^{1/2} = \omega \, |\chi_\infty|^{1/2} \, (1 + \eta^2)^{1/4} \qquad (38)$$

Upon squaring both sides Eq. (36) can be reduced to the following algebraic equation for η

$$1 + \left(\eta - \omega \frac{\chi_v}{\chi_\infty}\right)^2 = \frac{\omega^2/|\chi_\infty|^3}{(1 + \eta^2)^{1/2}} \tag{39}$$

Although Eq. (39) is of sixth order in η so that an analytical solution is not available, a graphical interpretation of Eq. (39) does provide an insight into the behavior of η. Let

$$y_1 = 1 + \left(\eta - \omega \frac{\chi_v}{\chi_\infty}\right)^2 \quad ; \quad y_2 = \frac{\omega^2/|\chi_\infty|^3}{(1 + \eta^2)^{1/2}}$$

then solutions of Eq. (39) are the intersections of the curves $y_1(\eta)$ and $y_2(\eta)$ in the y-η plane as shown in Fig. 3. As long as $\omega^2/|\chi_\infty|^3 > 1.0$, Eq. (39) will have two solutions η_1 and η_2 such that $\eta_1 < \omega \chi_v/\chi_\infty < \eta_2$, and from Eq. (37) it then follows that $K < 0$ when $\eta = \eta_1$ and $K > 0$ when $\eta = \eta_2$. Since K is to be evaluated at the accelerating sonic point, η_2 must be the appropriate solution of Eq. (39).

Although Eq. (39) cannot be solved for η as a function of χ_v and χ_∞, the equation can be solved for χ_v and a function of η and χ_∞ so that

$$\chi_v = \frac{\chi_\infty}{\omega}\left[\eta \mp \left\{\frac{\omega^2/|\chi_\infty|^3}{(1 + \eta^2)^{1/2}} - 1\right\}^{1/2}\right] \tag{40}$$

and the minus sign must be chosen in Eq. (40) to ensure that $K > 0$. From Eq. (40) it follows that $\eta < [(\omega^4/|\chi_\infty|^6) - 1]^{1/2}$ if χ_v is to be real while

$|\eta| > |[(\omega^{4/3}/|\chi_\infty|^2) - 1]^{1/2}|$ if χ_v is to be positive. For purely subsonic flows such that $\phi_x(x, 0) < \chi_\infty$ for all x the present formulation loses its meaning.

The variation of η with χ_v and χ_∞ as determined from Eq. (40) is shown in Fig. 4, for subsonic free stream flow with $\chi_\infty > 0$, and with $\omega = 1.0$ for convenience in calculation. Figure 4 reflects the influence of compressive viscosity on the linear solution for ϕ. From the definitions of η (Eq. (37)) and q (Eq. (35)) it follows that

$$q = \frac{|\chi_\infty|^{1/2}}{\omega \chi_v^2} (1 + \eta^2)^{1/4} \tag{41}$$

and the variation of q with χ_v and χ_∞ is shown in Fig. 5. Remarkably, q varies only slightly with χ_∞ but is mainly a function of the viscous parameter χ_v. For a sonic free stream with $\chi_\infty = 0$, Eq. (36) reduces to a cubic equation for K with solution

$$K = \left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\chi_v^3}{27}}\right]^{1/3} + \left[\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{\chi_v^3}{27}}\right]^{1/3}$$

while the Eq. (41) for q becomes

$$q = \frac{(\chi_v + K)^{1/2}}{\chi_v^2} \cong \frac{1 + \frac{2}{3}\chi_v}{\chi_v^2} \quad \text{when} \quad \chi_v \ll 1 \tag{42}$$

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From Eq. (36) and (37) it can be shown that

$$x^* = \frac{1}{\omega}\left[\frac{1}{2}\tan^{-1}\eta + \tan^{-1}\frac{\chi_\infty}{\chi_\infty\eta - \omega\chi_v}\right] \tag{43}$$

and the variation of the accelerating sonic point x^* with χ_v and χ_∞ is shown in Fig. 6 for a subsonic free stream with $\chi_\infty > 0$.

V. SOLUTION OF THE MATHIEU EQUATION

From Eq. (41) for q and the definition of χ_v and χ_∞ it can be shown that $q \to 0$ as $S \to 0$, that is as the amplitude of the wavy wall decreases to zero. Solutions of the Mathieu Eq. (35) in this limiting case $q = 0$ will be considered first for guidance to the appropriate choice of the constants of integration. Then

$$T = A_2 e^{\mu\xi} + A_3 e^{-\mu\xi} \tag{44}$$

where A_2 and A_3 are constants of integration and

$$\mu = \sqrt{-\hat{a}} = \frac{\sqrt{\chi_\infty^2 - 2A_1}}{\omega \chi_v}$$

From Eq. (34) it then follows that

$$\zeta = u^{(1)}(x, 0) - \chi_\infty = \frac{-\omega \chi_v \mu \left[(A_2/A_3) e^{\mu\xi} - e^{-\mu\xi}\right]}{(A_2/A_3) e^{\mu\xi} - e^{-\mu\xi}} \tag{45}$$

and A_2/A_3 is now the only independent constant of integration. Letting $A_2/A_3 \to \infty$

$$\zeta = -\omega \chi_v \mu = -\sqrt{\chi_\infty^2 - 2A_1}$$

and choosing $A_1 = 0$ yields

$$\zeta = -\chi_\infty \tag{46}$$

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corresponding to undisturbed free stream flow. This is the expected result in the limiting case of vanishing amplitude. However, with $A_1 = 0$ three other possible solutions are

$$(A_2/A_3) = 1.0 \qquad ; \qquad \zeta = - \chi_\infty \tanh \frac{\chi_\infty}{\omega \chi_v} \xi \qquad (47)$$

$$(A_2/A_3) = 0 \qquad ; \qquad \zeta = + \chi_\infty \qquad (48)$$

$$(A_2/A_3) = - 1.0 \qquad ; \qquad \zeta = - \chi_\infty \coth \frac{\chi_\infty}{\omega \chi_v} \xi \qquad (49)$$

The solution (47) corresponds to Taylor's (1910) solution for the structure of a weak normal shock wave while the solution (49) diverges at $\xi = 0$. None of the solutions (47)-(49) are appropriate limiting solutions for vanishing amplitude.

When $q \neq 0$, it follows from the Floquet theory for second order linear equations with periodic coefficients that solutions of the Mathieu equation can be written in the form (Abramowitz and Stegun, 1964)

$$T = A_2 e^{-\mu\xi} P(\xi) + A_3 e^{\mu\xi} P(-\xi) \qquad (50)$$

$P(\xi)$ is a function with period π or 2π and $\mu = \mu(\hat{a}, q)$ is the characteristic exponent which may be real, zero, imaginary, or complex, and is a function of the parameters $\hat{a}$ and q. With $\mu = 0$ one of the solutions T will be periodic of

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period π or 2π and for a given q the values of $\hat{a}$ resulting in $\mu = 0$ are called characteristic values of the Mathieu equation. With μ real T diverges at either $\xi \to +\infty$ or $\xi \to -\infty$ and the solution is called unstable, while if μ is imaginary T will be periodic but of period other than π or 2π and the solution is termed a stable solution. The $(\hat{a}, q)$ plane can thus be subdivided into regions of stable and unstable solutions separated by $\mu(\hat{a}, q) = 0$. The portion of the $(\hat{a}, q)$ plane pertinent here is shown qualitatively in Fig. 7 which also shows curves of constant μ.

With q = 0 it has already been shown that $\mu = \sqrt{-\hat{a}} = \chi_\infty / \omega \chi_v$ and that $\hat{a} < 0$ so that the solution for T lies in the unstable region along the negative $\hat{a}$ axis (Fig. 7). When q > 0 it is to be expected that the solution will, at least for some range of q, still lie in this unstable region, however $\mu \neq \sqrt{-\hat{a}}$ but will depend on both q and $\hat{a}$. Then the solution (50) can be expressed in the form (McLachlan, 1947)

$$T = A_2 e^{\mu\xi} \sum_{r=-\infty}^{\infty} C_{2r} e^{2r\xi i} + A_3 e^{-\mu\xi} \sum_{r=-\infty}^{\infty} C_{2r} e^{-2r\xi i} \qquad (51)$$

with the recurrence relation

$$[\hat{a} - (2r - i\mu)^2] C_{2r} - q(C_{2r+2} + C_{2r-2}) = 0 \qquad (52)$$

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From Eq. (52) it follows that C_{2r} and C_{-2r} are complex conjugates. Then letting

$$C_{2r} = \rho_{2r} \, e^{i\phi_{2r}} \quad , \quad C_{-2r} = \rho_{2r} \, e^{-i\phi_{2r}} \quad , \quad C_0 = 2\rho_0$$

Equations (34) and (51) yield the following solution for ζ, the deviation of the velocity $u^{(1)}$ at the wall from the sonic value:

$$\frac{\zeta}{\omega\chi_v} = \frac{\dfrac{A_2}{A_3} e^{\mu\xi}\left[-\mu\rho_0 + \sum_{r=1}^{\infty}\left\{2r\rho_{2r}\sin(2r\xi+\phi_{2r}) - \mu\rho_{2r}\cos(2r\xi+\phi_{2r})\right\}\right]}{\dfrac{A_2}{A_3} e^{\mu\xi}\left[\rho_0 + \sum_{r=1}^{\infty}\rho_{2r}\cos(2r\xi+\phi_{2r})\right] + e^{-\mu\xi}\left[\rho_0 + \sum_{r=1}^{\infty}\rho_{2r}\cos(2r\xi-\phi_{2r})\right]}$$

$$+ \frac{e^{-\mu\xi}\left[+\mu\rho_0 + \sum_{r=1}^{\infty}\left\{2r\rho_{2r}\sin(2r\xi-\phi_{2r}) + \mu\rho_{2r}\cos(2r\xi-\phi_{2r})\right\}\right]}{}$$

$$\tag{53}$$

Letting $A_2/A_3 \to \infty$ as in the limiting $q = 0$ case considered above the solution (53) for ζ becomes

$$\zeta = -\omega\chi_v\mu + \frac{\omega\chi_v\displaystyle\sum_{r=1}^{\infty} 2r\rho_{2r}\sin(2r\xi+\phi_{2r})}{\rho_0 + \displaystyle\sum_{r=1}^{\infty}\rho_{2r}\cos(2r\xi+\phi_{2r})} \tag{54}$$

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From the recurrence relation (52) it follows that as $q \to 0$, C_{2r} and hence $\rho_{2r} \to 0$ so that $\zeta \to -\omega \chi_v \mu$ which is identical to the limiting solution (46). The solution (54) has period π in ξ or is of period 2π in $\omega \chi$, that is the same period as the wavy wall.

If $A_2/A_3 = 0$ the solution reduces to Eq. (48) in the limit $q \to 0$. When $A_2/A_3 = 1.0$ the solution for $\int$ behaves like a weak normal shock wave at the origin with a superimposed oscillation, while the solution will diverge at some value of ξ when $A_2/A_3 < 0$. Thus, the choice $A_2/A_3 \to \infty$ appears to be the only proper one for the wavy wall problem.

The constant of integration A_1 determines the parameter $\hat{a}$ (Eq. (35)) and hence influences the solution through both the recurrence relation (52) and through the dependence of the characteristic exponent μ upon $\hat{a}$. With $q \neq 0$ the choice of A_1 is thus not as straightforward as in the limiting $q = 0$ case. A property of Hosokawa's (1960) inviscid analysis but not of the present viscous solution is that the correction g_x vanishes at the accelerating sonic point defined by Eq. (28). In the viscous case $g_x(x^*)$ will depend upon the integration constant A_1; therefore, in order to compare the viscous and inviscid results A_1 has been chosen to make $g_x(x^*) = 0$. Therefore, at the accelerating sonic point, $x = x^*$ or $\xi = \xi^*$, where $\zeta(\xi^*) = 0$, the condition

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$$\mu(\hat{a}, q) = \frac{\displaystyle\sum_{r=1}^{\infty} 2r\rho_{2r} \sin(2\xi^* + \phi_{2r})}{\rho_0 + \displaystyle\sum_{r=1}^{\infty} \rho_{2r} \cos(2r\xi^* + \phi_{2r})} \tag{55}$$

must be satisfied. Equation (55) is an implicit equation for $\hat{a}$ and hence A_1, which can only be solved by trial and error.

The series in the solution (54) converge very rapidly so that only three or four terms of the series need to be retained in the numerical evaluation of ζ. For low values of q the values of μ for different values of $\hat{a}$ and q have been obtained from graphs in Abramowitz and Stegun (1964). For large values of q asymptotic expressions given by Erdélyi et al (1955) have been used to determine μ as described in Appendix A. In practice computations are carried out by first choosing a value of q, which, from Fig. 5, is equivalent to fixing the viscous similarity parameter χ_v. ζ is then computed for arbitrary values of $\hat{a}$, and the position of the sonic point, ξ^* read from these solutions can be used to determine χ_∞ from Fig. 6. The computation of ζ can then be repeated using values of $\hat{a}$ corresponding to equal increments in χ_∞.

VI. RESULTS AND DISCUSSION

The variation of ζ with χ_∞ has been determined for $\chi_v = 0.61$ and 0.33 as shown in Figs. 8 and 9. Hosokawa's (1960a) inviscid solution for several different values of χ_∞, described in Appendix B, is shown in Fig. 10. In the case of a sonic free stream with $\chi_\infty = 0$ the variation of ζ is plotted in Fig. 11 for $\chi_v = 0.61$, 0.48, 0.33, 0.27, and for $\chi_v = 0$ corresponding to Hosokawa's (1960a) inviscid solution.

From Figs. 8 and 9 it can be seen that in the viscous theory shock discontinuities terminating regions of supersonic flow are replaced by smooth compressions across which the Rankine-Hugoniot conditions are not necessarily satisfied. Comparison of Figs. 8 and 9 shows the influence of decreasing χ_v or increasing Re upon the solutions. It can be seen that the transition to subsonic flow becomes steeper with decreasing χ_v. With a sonic free stream $\chi_\infty = 0$, χ_v does not seem to affect the location of the compression wave; in fact, the inflection points of the sonic compressions occur at $x = 1.25\,\pi$ which is also the location of the corresponding shock discontinuity in the inviscid solution (Fig. 10). With a subsonic free stream, $\chi_\infty > 0$, on the other hand the effect of viscosity is to shift the compressions upstream and to shorten the supersonic region as compared to the inviscid solution of Hosokawa (Fig. 10). Comparison of the viscous and inviscid solutions shows that viscosity has a negligible effect on the accelerating portion of the flow preceding the shock transition.

Figure 11 shows that the viscous solution approaches the inviscid solution as $\chi_v \to 0$ when the free stream flow is sonic ($\chi_\infty = 0$); however, as indicated above this no longer appears to be true when the free stream is subsonic ($\chi_\infty > 0$). This interesting result, which suggests that it may be essential to include viscous effects in the analysis of certain transonic flows, bears further investigation. Since the theory developed here rests on a number of approximations and assumptions, the result that the subsonic solutions fail to approach the inviscid solutions as $\chi_v \to 0$ can only be considered tentative.

The sonic solutions have certain special features. For these solutions the characteristic exponent $\mu = 0$ so that the sonic solutions correspond to the characteristic solutions of the Mathieu equation. Further, with $\chi_\infty = 0$, the phase angle β, which arises in the linear solution for ϕ has a value of $\pi/2$ just as in the inviscid case while $x^* = \pi/4$ and is independent of χ_v.

The choice of the condition $g_x(x^*) = 0$, used here, will have a particularly important influence on the behavior of the subsonic solutions. Setting $g_x(x^*) = 0$ is based in part on the inference of Oswatitsch (1955) that the linearized solution should be valid near the accelerating sonic point, and of course this is also the point at which the constant K coincides with the actual acceleration ϕ_{xx} of the linearized flow; however, there is no rigorous justification for this choice. $g_x(x^*)$ could also, for example, be chosen to make the inflection points of the compressions in the viscous solutions coincide with the position of the inviscid

shock discontinuities. With this choice all the viscous solutions would probably approach the inviscid solutions as $\chi_v \to 0$.

The flows considered here correspond to low Reynolds numbers. For example, for a body with thickness ratio $S = 0.01$ the range $0.27 < \chi_v < 0.61$ corresponds to $64 > Re > 28$, and $0.016 < (L_v/L) < 0.035$ (Fig. 2(b)). For this value of S, M_∞ will be very close to unity and the range $0 < \chi_\infty < 0.8$ corresponds to $1 > M_\infty > 0.970$. Boundary layers have, of course, been neglected in the present analysis, and the Re is based upon the wavelength or characteristic body length. In an actual flow the behavior of the free stream shock wave will also be influenced by disturbances induced in the boundary layer by the rapid free stream compression. Then an appropriate Reynolds number to describe the behavior of the flow near the shock wave might be more appropriately based upon a length characteristic of the boundary layer thickness rather than the length of the body. It is thus possible even when the Reynolds number based on $\overline{L}$ is extremely large, that the local Reynolds number governing the shock behavior may be extremely small, that is of the same order as the Reynolds numbers considered here.

As compared to the previous paper on wavy wall flow (Sichel and Yin 1969) it has now been possible to obtain solutions in analytical form. The ratio of wall amplitude to wave length and to a viscous length appears here only through the parameters χ_v and χ_∞ whereas these ratios were introduced through the

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boundary conditions in the previous paper. It is difficult to compare the present results to those obtained by numerical integration in the previous paper although there seems to be some difference. In particular, with a sonic free stream the location of the inviscid shock and the viscous compression do not coincide as in the present paper. As pointed out in Section 5 above there are several solutions for ζ which are periodic but only one has the proper limiting behavior as the wall amplitude $S \rightarrow 0$. While it is easy to choose the appropriate solution here, the difficulty of discriminating between the different solution branches when numerical integration is used may account for the differences between the previous and present paper.

Solutions have been obtained only for sonic and subsonic flow, i.e. for $\chi_\infty \geq 0$, in the present paper. However the present analysis can be readily used to evaluate solutions for supersonic free stream flows with $\chi_\infty < 0$, by letting the constant of integration $A_2/A_3 = 0$ in Eq. (53), and by extending the range of the calculations for the relation between χ_v, χ_∞, and β. In the limit $\chi_\infty \rightarrow 0$ the supersonic solutions will approach the same characteristic Mathieu solution as the subsonic solutions.

The present results provide an indication of the role of χ_v and χ_∞ in viscous transonic flow. The boundary layer has been neglected and since experimental results for transonic wavy wall flow are not available there is no way of comparing theory and experiment.
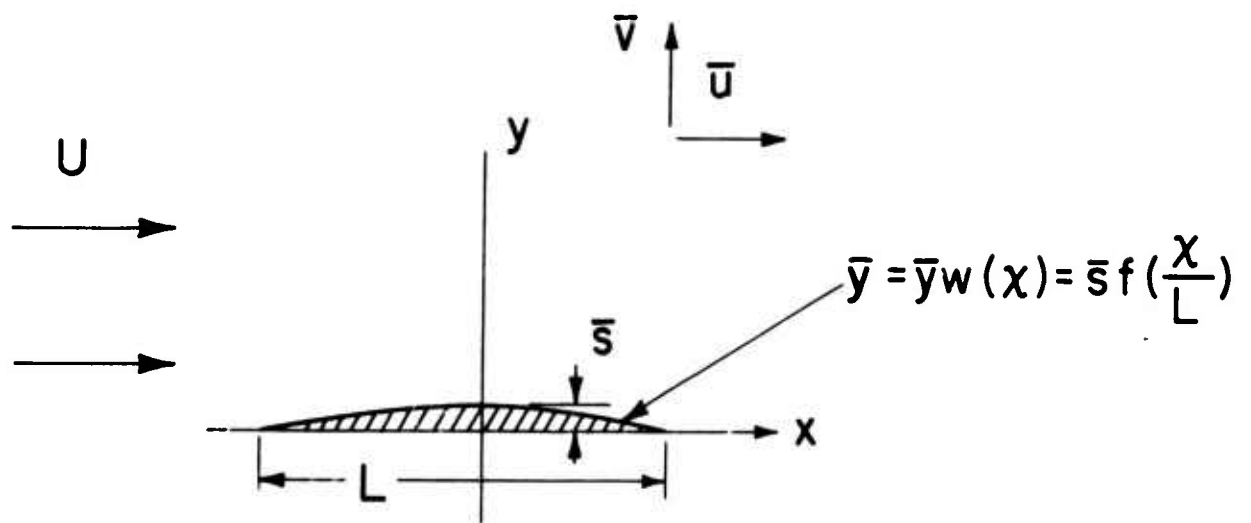
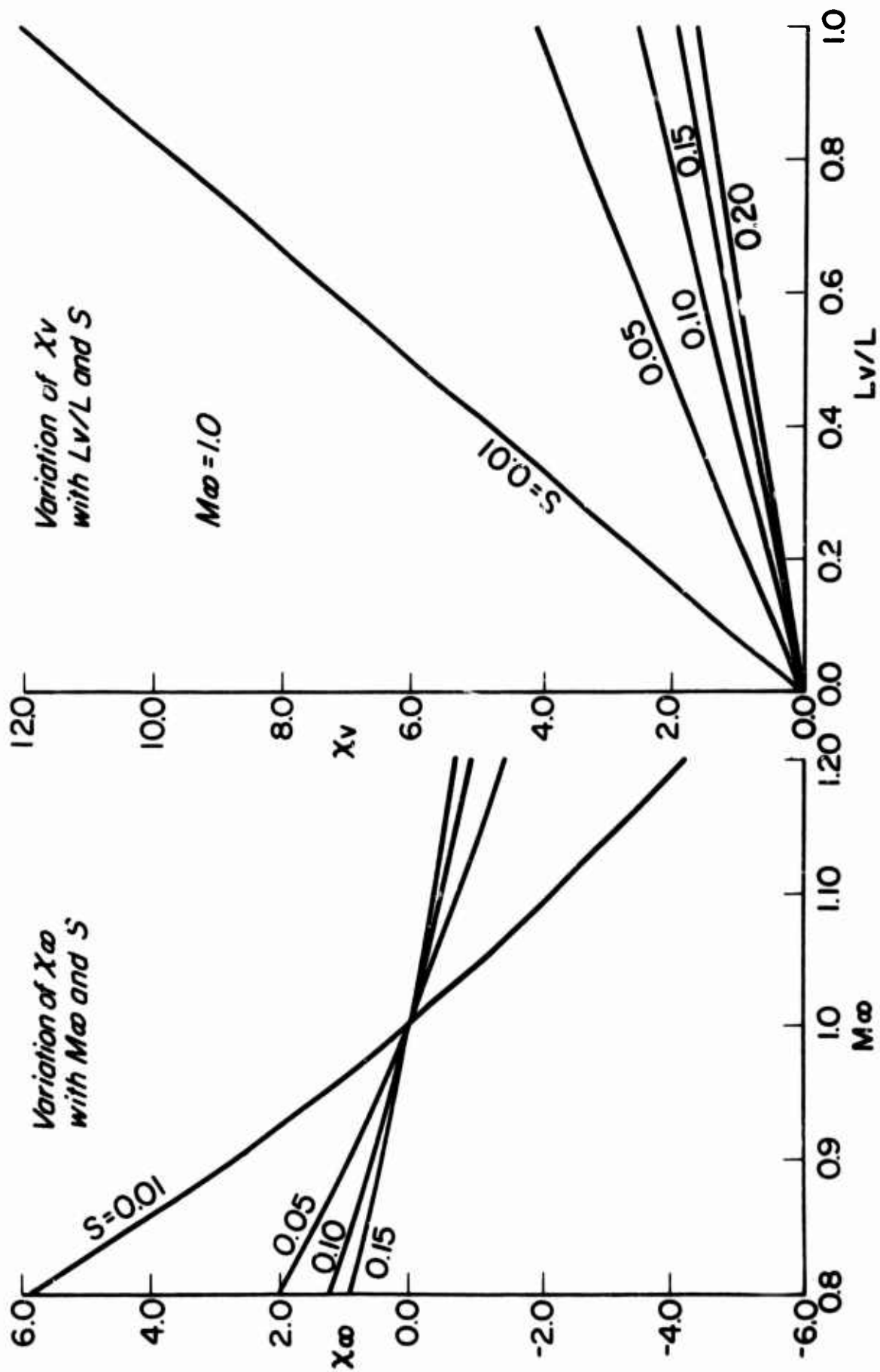Figure 1. Definition of the Coordinate System.

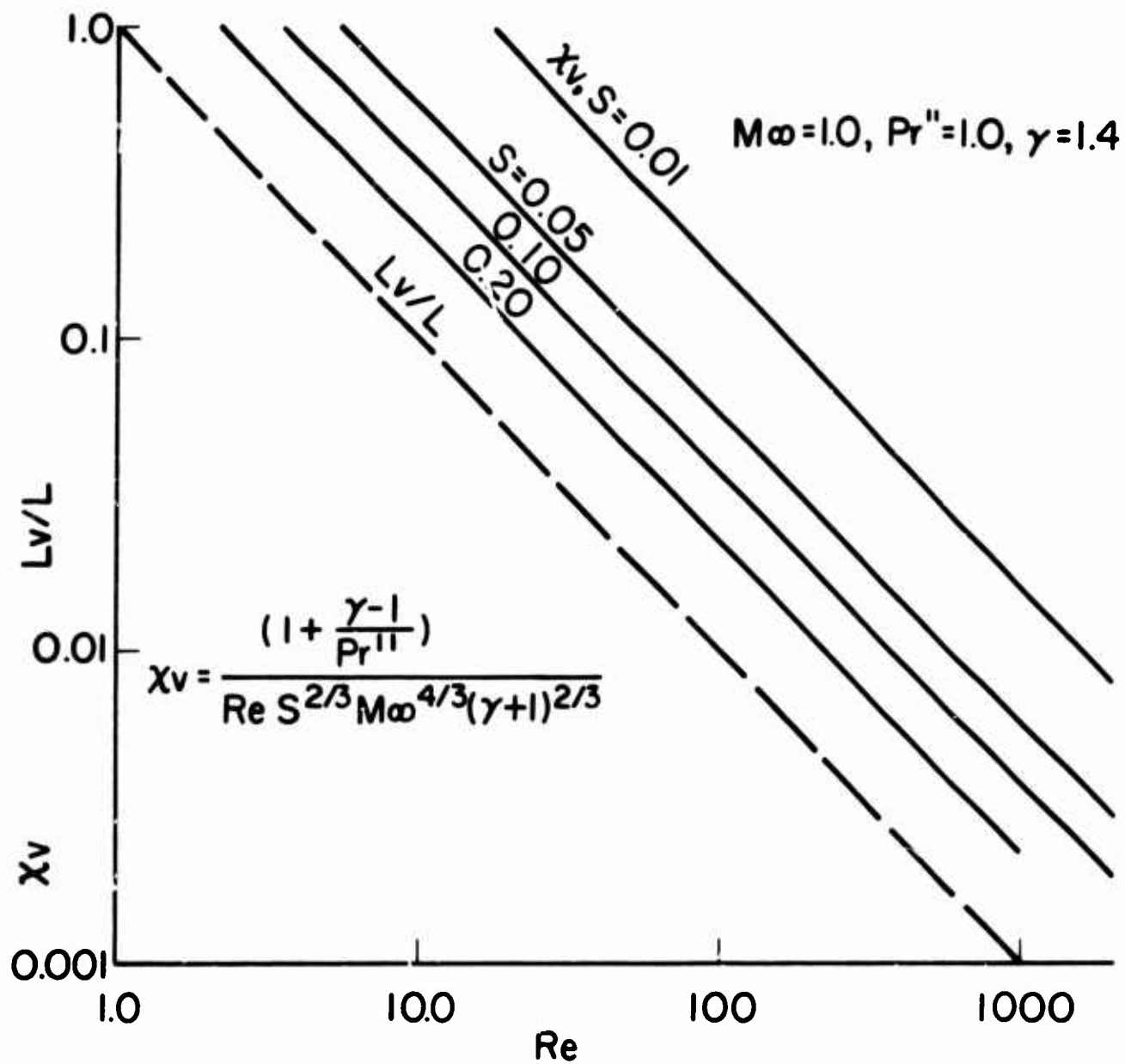Figure 2a. The Variation of χ_v, χ_∞, and L_v/L with S and M_∞.

The graph shows lines with labels:

X_v, $S=0.01$

$S=0.05$

0.10

0.20

L_v/L

$M\infty = 1.0$, $Pr'' = 1.0$, $\gamma = 1.4$

$$X_v = \frac{\left(1 + \frac{\gamma-1}{Pr''}\right)}{Re\, S^{2/3} M\infty^{4/3} (\gamma+1)^{2/3}}$$

Vertical axis: L_v/L (values 1.0, 0.1, 0.01, 0.001) and X_v

Horizontal axis: Re (values 1.0, 10.0, 100, 1000)

Figure 2b.   The Variation of X_v and L_v/L with the Reynolds Number.

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Variation of y_1 and y_2 with η
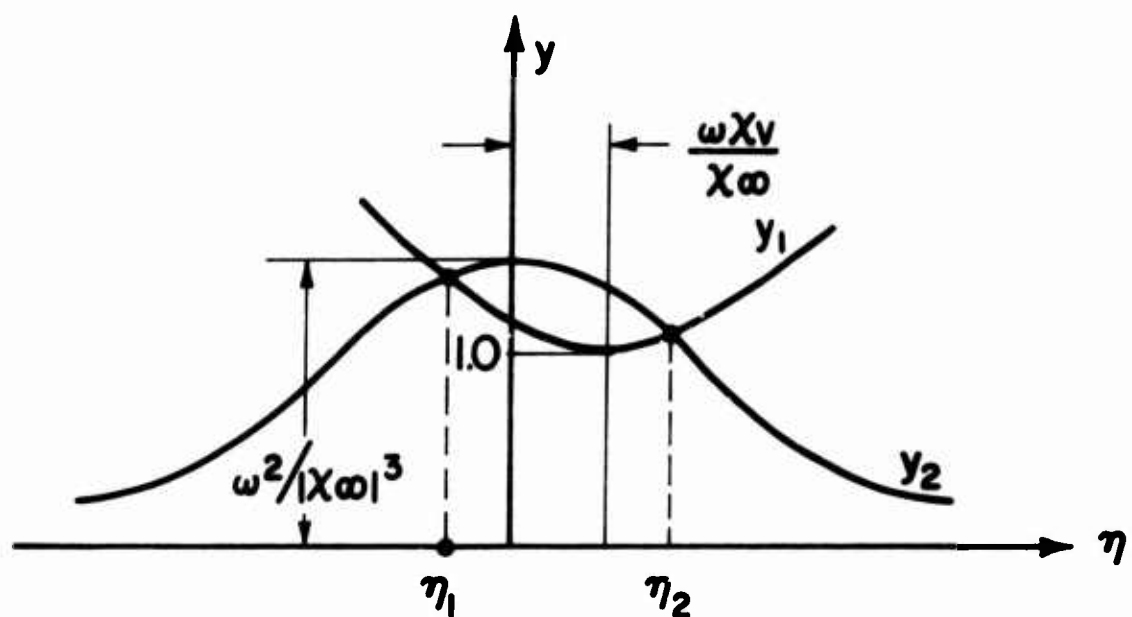
Figure 3.   Graphical Solution for η.

Variation of X_V with η and X_∞

Figure 4.

Figure 5.

Variation of q with X_∞ and X_v

$X_\infty = 0.0, 0.1$

$X_\infty = 0.9$

$X_\infty = 0.5$

X_v

Figure 6.

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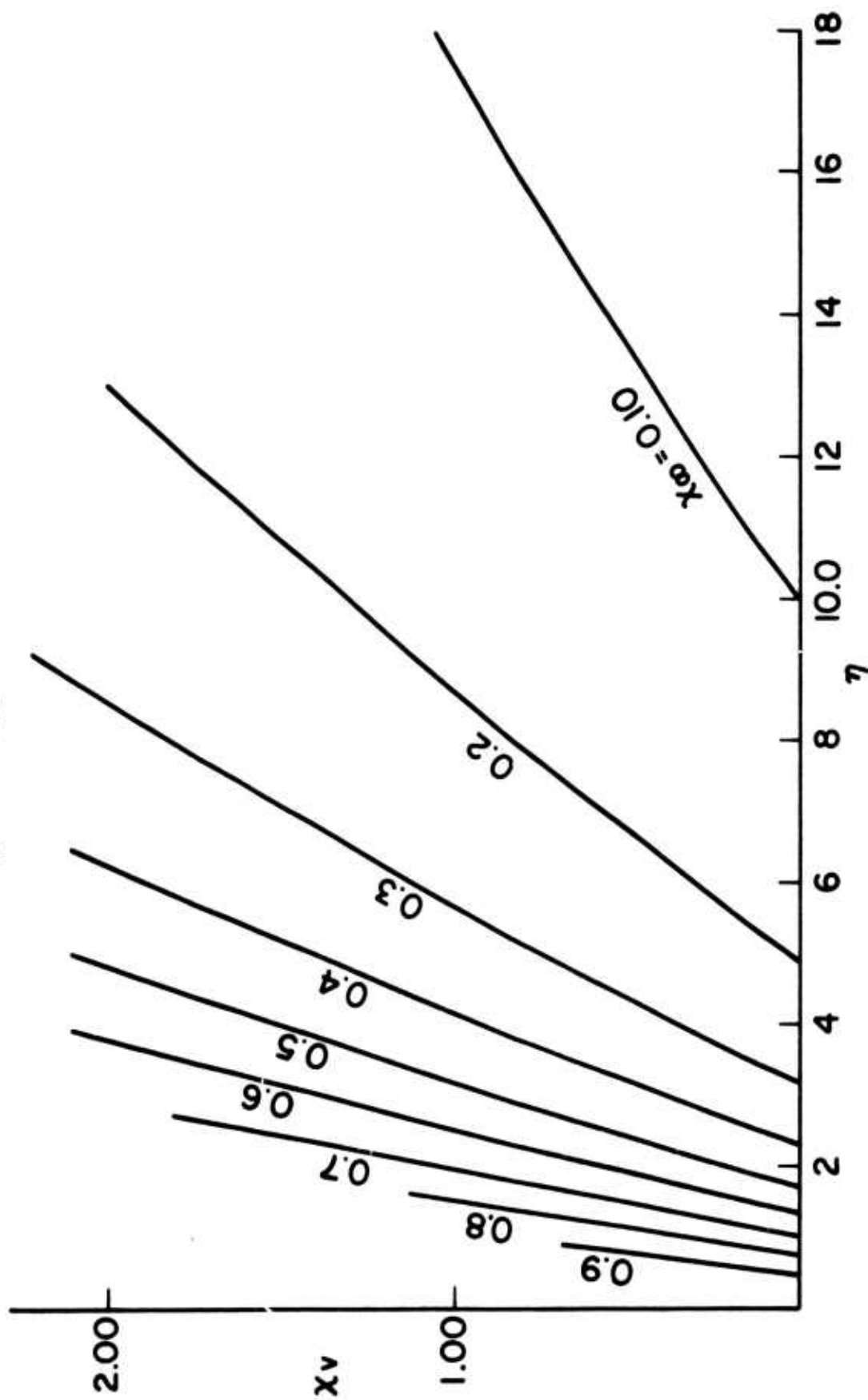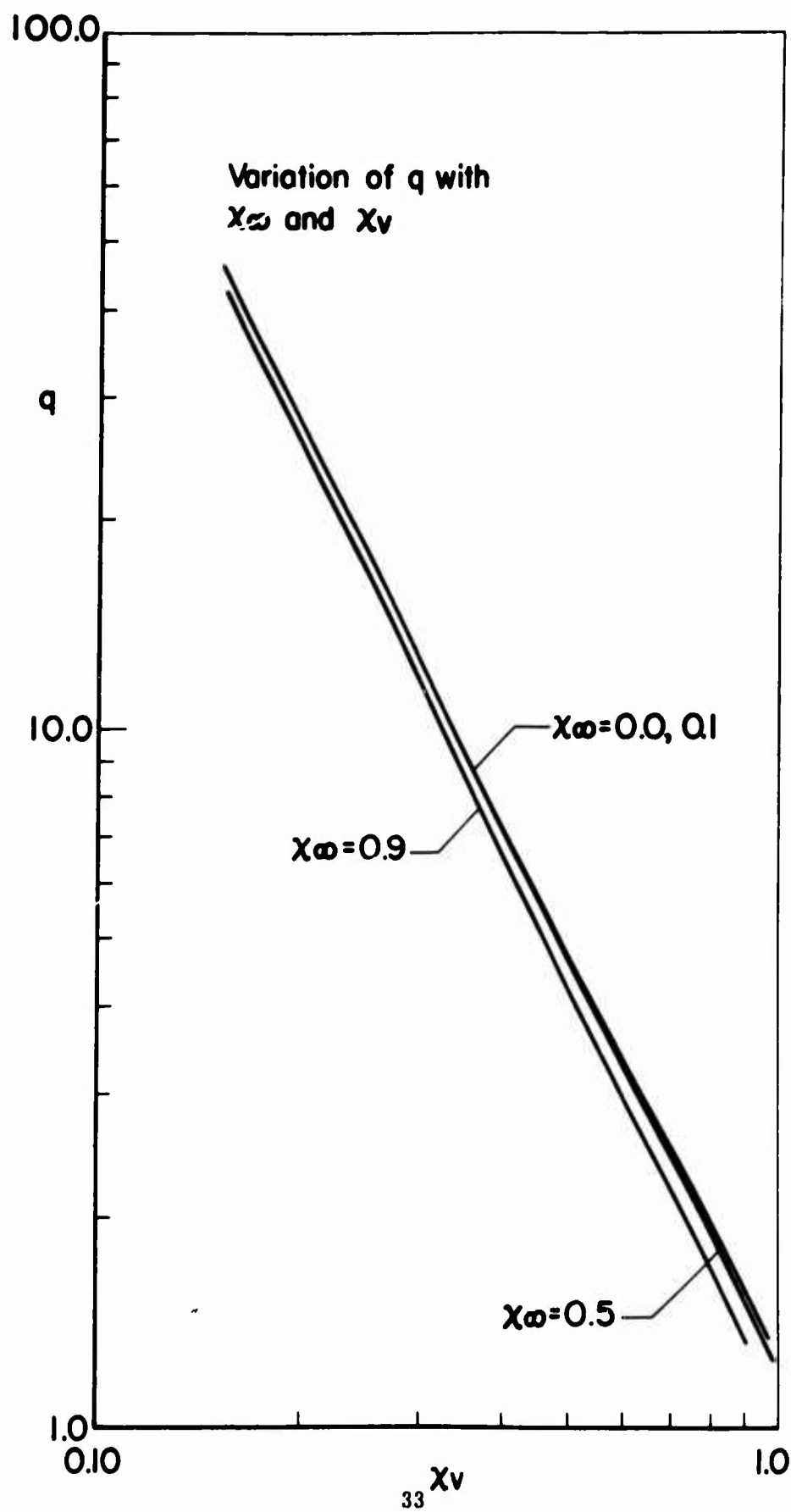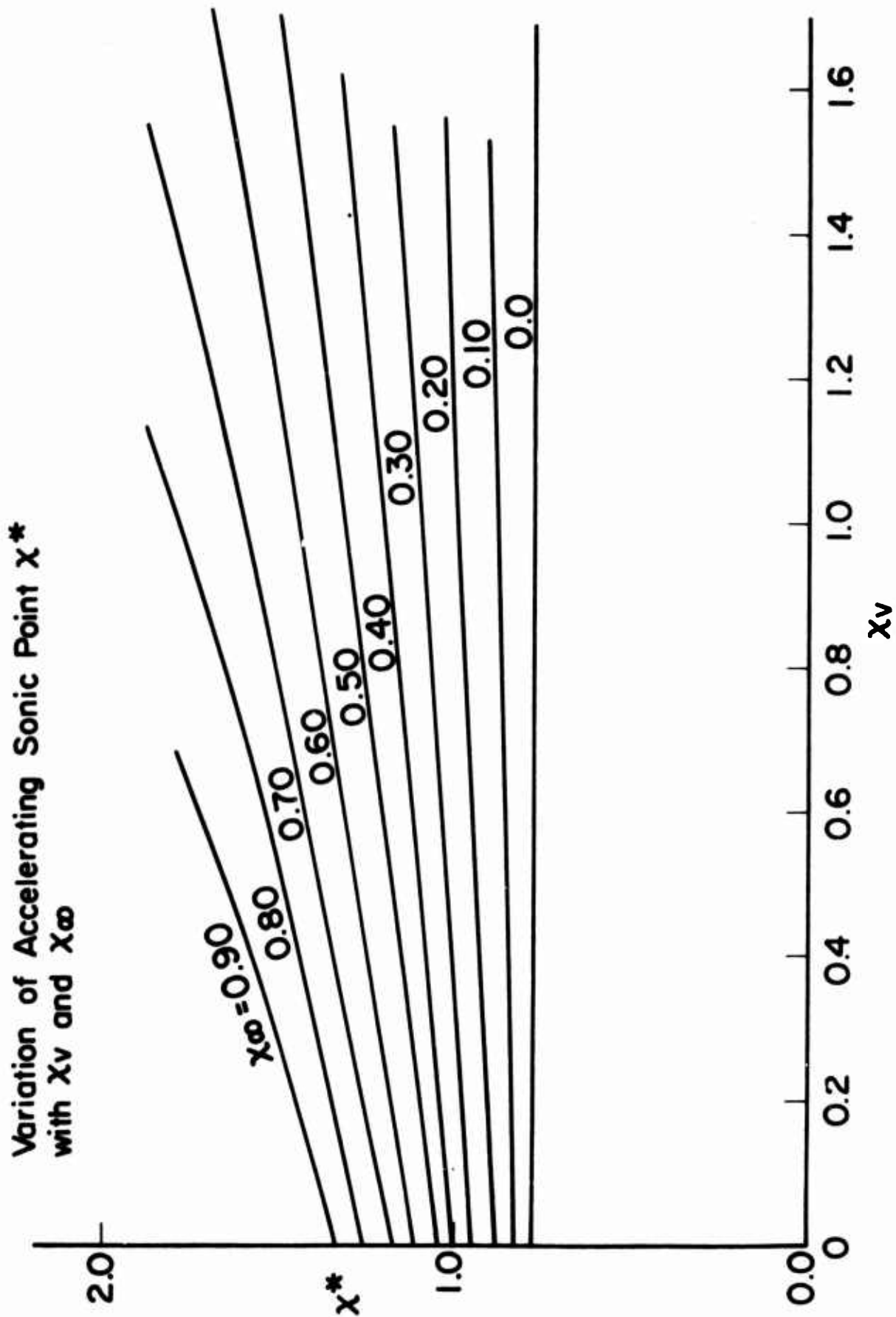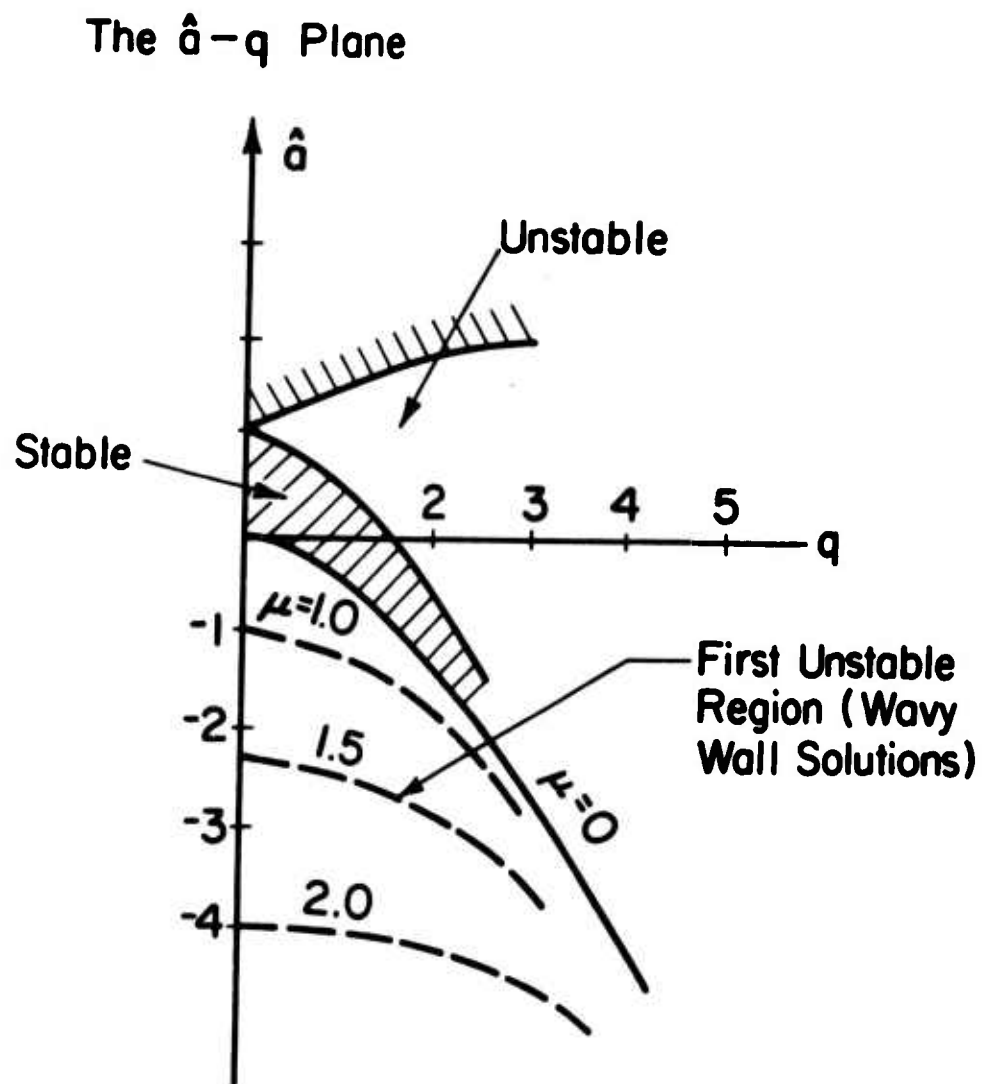
The â-q Plane

Figure 7. Stable and Unstable Regions in the $\hat{a}$ - q Plane.

Variation of $\zeta = u^{(1)} - u_c^{(1)}$ with X

$\omega = 1.00$
$X_v = 0.61$
$q = 3.0$

$X\omega = 0.0$
0.1
0.2
0.3

Figure 8.

Variation of $\zeta = u^{(1)} - u_c^{(1)}$ with X
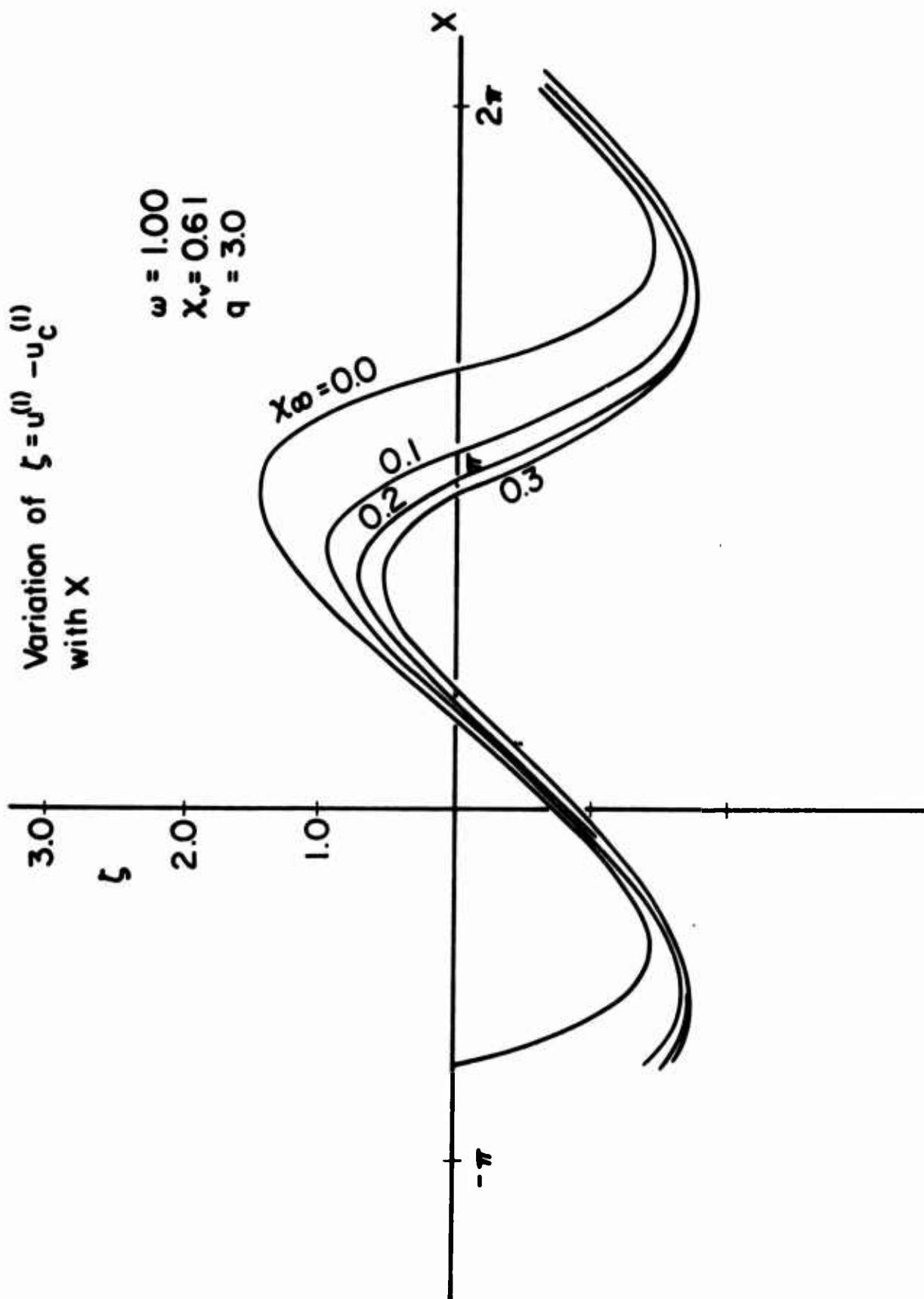
$\omega = 1.00$
$X_v = 0.33$
$q = 10.0$
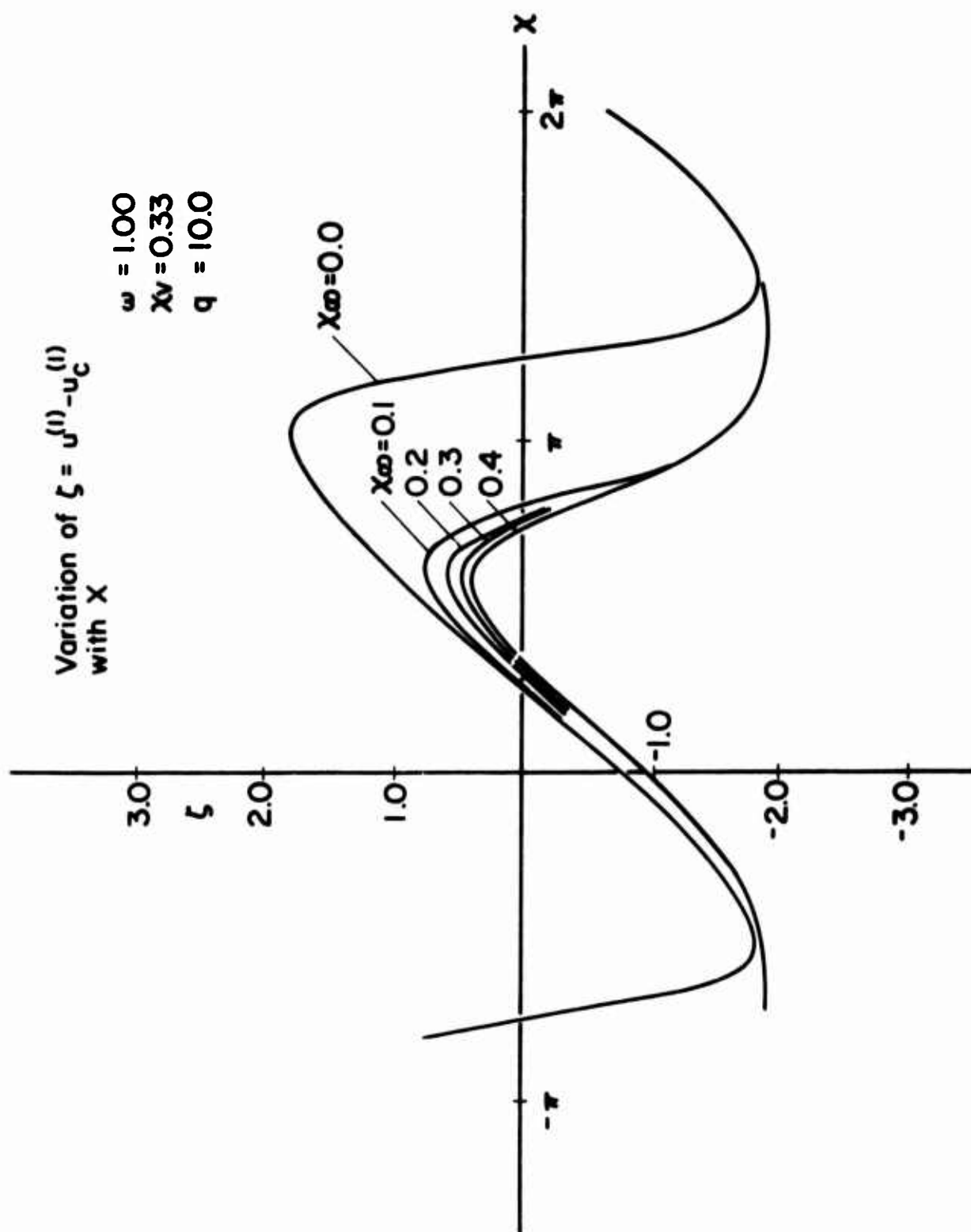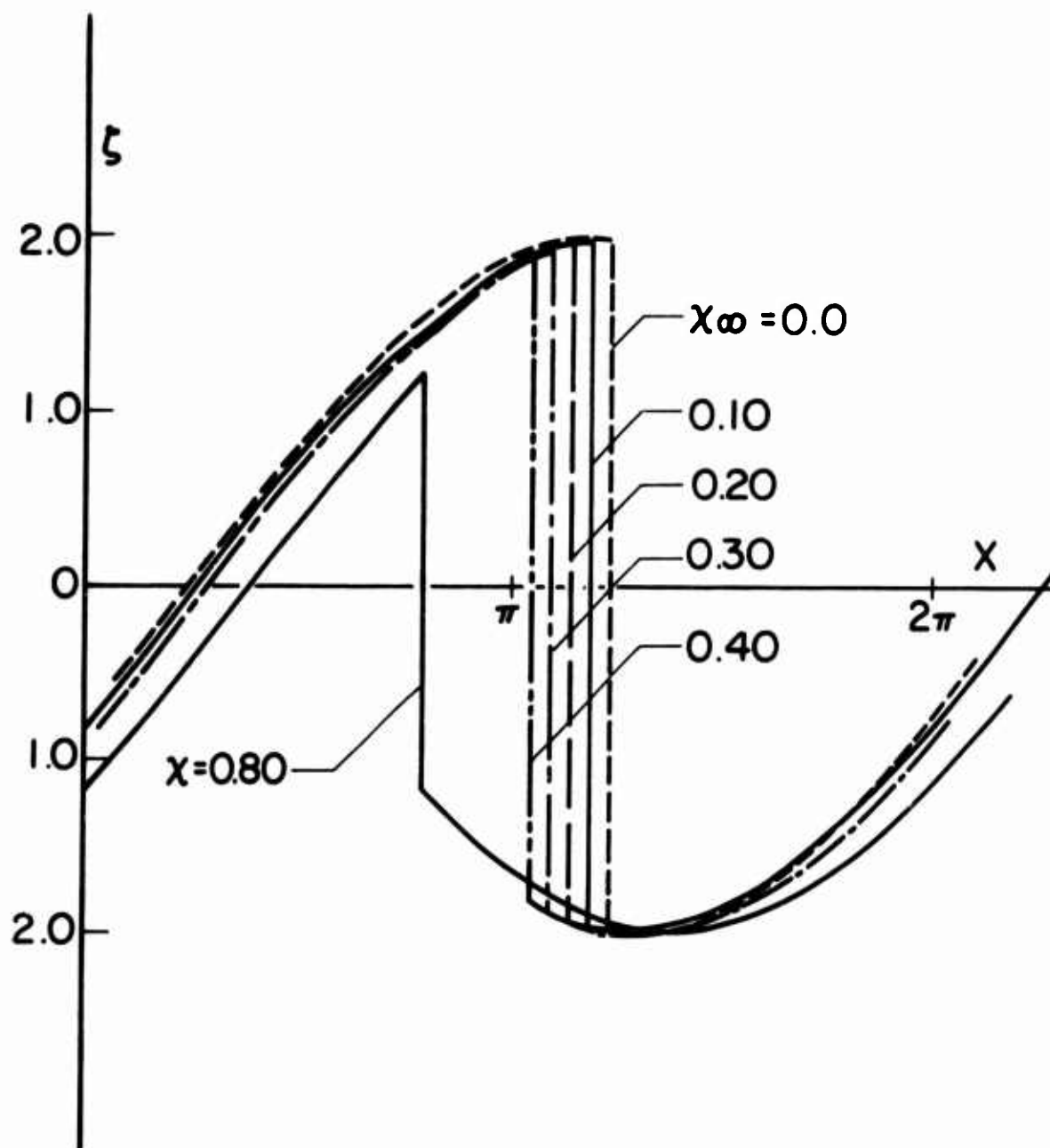
Figure 9.

Figure 10. The Variation of ζ with x; Hosokawa's Inviscid Solution.

Variation of $\zeta = u^{(1)} - u_c^{(1)}$ With X for Sonic Free Stream
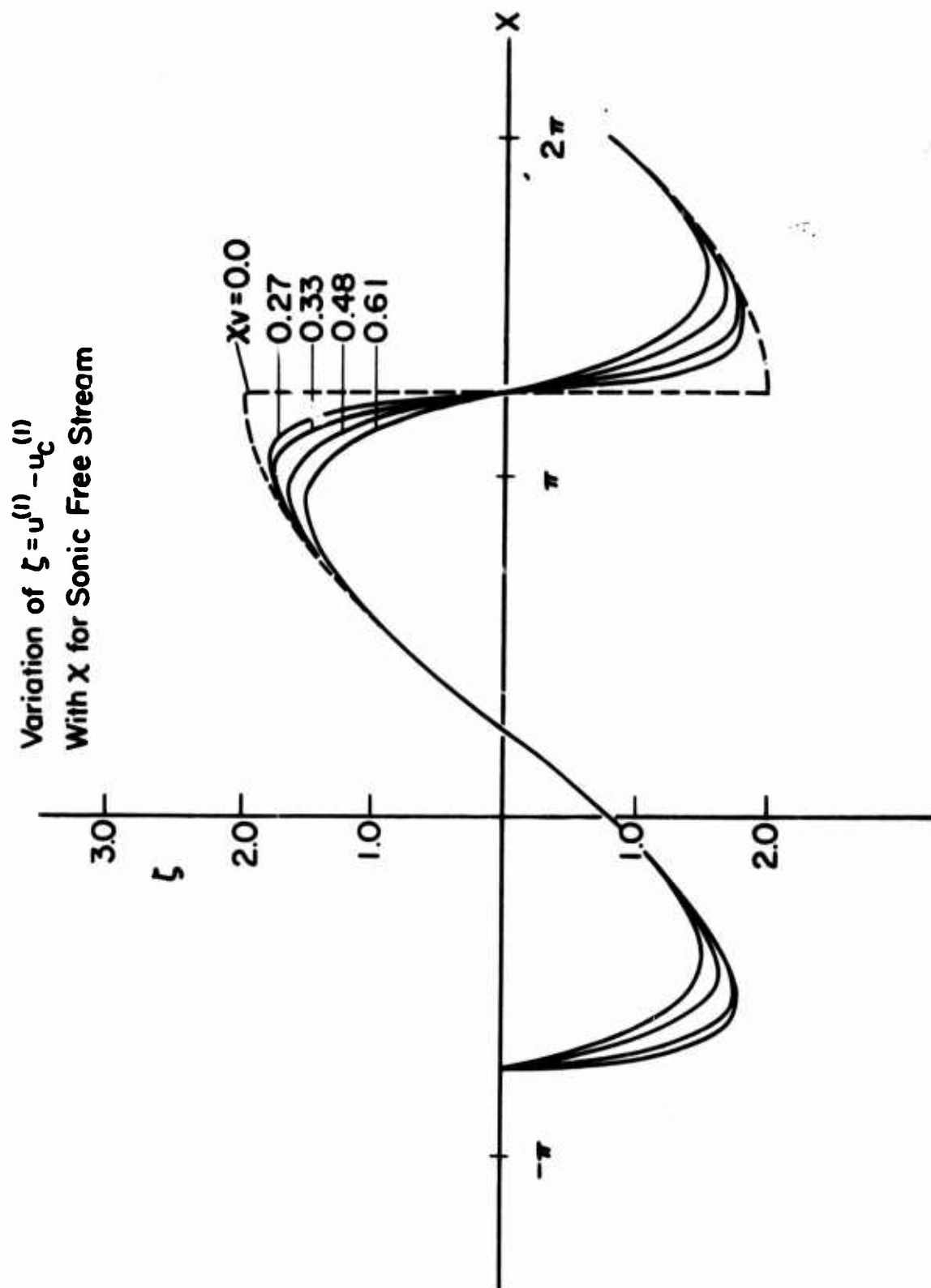
$Xv = 0.0$
0.27
0.33
0.48
0.61

Figure 11.

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APPENDIX A

For $q \gg 1$, $\hat{a} \gg 1$ the asymptotic expressions for the characteristic value μ given by Erdelyi et al (1955) can be expressed in terms of complete elliptic integrals. Since these expressions are not, to the author's knowledge, available elsewhere they are presented below.

For $\hat{a} < -2\,|q|$ the expression for μ can be reduced to

$$\mu = (2/\pi)\ \sqrt{2q - \hat{a}}\ E\left(\frac{4q}{2q - \hat{a}}\right) \tag{A-1}$$

where $E(\ )$ is a complete elliptic integral of the second kind (Abramowitz and Stegun, 1964). On the other hand for $-2q < \hat{a} < 2q$

$$\cosh \mu\pi = \cos I_1\, \cosh I_2 \tag{A-2}$$

where

$$I_1 = 2\ \sqrt{2q + a}\, [\,\tilde{m}^{1/2}\, E(1/\tilde{m}) - (\tilde{m} - 1)\ \tilde{m}^{-1/2}\, K(1/\tilde{m})]$$

$$\tilde{m} = 4q/(2q + a)$$

$$I_2 = 2\ \sqrt{2q - a}\, [\,m^{1/2}\, E(1/m) - (m - 1)\ m^{-1/2}\, K(1/m)]$$

$$m = 4q/(2q - a)$$

In these expressions $K(\ )$ is a complete elliptic integral of the first kind.

APPENDIX B

Hosokawa's (1960b) inviscid solution when expressed in the variables used here becomes

$$\zeta = \pm \sqrt{2\chi_\infty^2 + 2K\left[\phi - \phi(x^*)\right] - 2\chi_\infty \phi_x} \qquad (B-1)$$

with the positive sign when $\phi_x > \chi_\infty$ and the negative sign when $\phi_x < \chi_\infty$. In the inviscid case

$$\phi = -\omega^{-1/3} \cos\left(\omega x - \frac{1}{2}\beta\right)$$

$$K = \omega \chi_\infty \left[(\omega^{4/3}/\chi_\infty^2) - 1\right]^{1/2}$$

$$\phi(x^*) = -(\chi_\infty/\omega)\left[(\omega^{4/3}/\chi_\infty^2) - 1\right]^{1/2}$$

$$\beta = \arctan\left[(\omega^{4/3}/\chi_\infty^2) - 1\right]^{1/2}$$

DOCUMENT CONTROL DATA · R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION
Department of Aerospace Engineering The University of Michigan Ann Arbor, Michigan 48104	Unclassified
	2b. GROUP

3. REPORT TITLE

Solution of the Viscous Transonic Equation for Flow Past a Wavy Wall

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Interim Technical Report

5. AUTHOR(S) *(First name, middle initial, last name)*

Martin Sichel

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1970	42	11

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
DAHC04-68-C-0008	
b. PROJECT NO.	01361-1-T
c.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
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13. ABSTRACT

The viscous transonic small disturbance equation has been recast into a form which displays the role of the usual inviscid transonic similarity parameter and of a newly defined viscous transonic similarity parameter in the two dimensional flow past slender bodies. Using the approximate method of Hosokawa the solution for viscous transonic flow past a wavy wall has been obtained in analytical form, and displays the role of the above similarity parameters. The influence of the Reynolds number and free stream Mach number on the supersonic pockets which arise at the wall surface has been investigated.

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